

Gradient vector fields on cosymplectic manifolds

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1992 J. Phys. A: Math. Gen. 25 175

(<http://iopscience.iop.org/0305-4470/25/1/022>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.58

The article was downloaded on 01/06/2010 at 16:26

Please note that [terms and conditions apply](#).

Gradient vector fields on cosymplectic manifolds

F Cantrijn†‡, M de León§ and E A Lacomba||

† Instituut voor Theoretische Mechanica, Rijksuniversiteit Gent, Krijgslaan 281-S9, B-9000 Gent, Belgium

§ Unidad de Matemáticas, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain

|| Departamento de Matemáticas, Universidad Autónoma Metropolitana, Iztapalapa, PO Box 55-543, Mexico DF, Mexico

Received 26 June 1991

Abstract. In this paper we study some geometrical properties of gradient vector fields on cosymplectic manifolds, thereby emphasizing the close analogy with Hamiltonian systems on symplectic manifolds. It is shown that gradient vector fields and, more generally, local gradient vector fields can be characterized in terms of Lagrangian submanifolds of the tangent bundle with respect to an induced symplectic structure. In addition, the symmetry and reduction properties of gradient vector fields are investigated.

1. Introduction

Canonical manifolds have been identified and studied by Lichnerowicz as the natural geometrical framework for the description of time-dependent mechanical systems (see, for example [1, 2] and, for a view on the physical applications, see also [3]). A canonical manifold is a Poisson manifold of constant rank, fibred over \mathbb{R} (the ‘time-axis’) and such that the connected components of the fibres are symplectic leaves of the Poisson structure. The Lie algebras associated with a canonical structure have been investigated by Flato *et al* [4]. Lie group actions on a canonical manifold and the concept of momentum map in this framework have been studied by Marle [5]. A particular class of canonical manifolds consists of those possessing a closed 2-form, the restriction of which to each fibre yields the symplectic form induced by the Poisson structure. Canonical manifolds with a closed 2-form correspond to what are elsewhere also called *cosymplectic manifolds* (cf [6]). In a recent paper, Albert [7] has generalized the Marsden–Weinstein reduction theory for symplectic manifolds with symmetry to cosymplectic manifolds and to general contact manifolds. In doing so he has conceived a unified approach to cosymplectic and contact structures by identifying them as limiting cases of a ‘transitive almost contact structure’.

In this paper we wish to focus our attention on a special class of vector fields on a cosymplectic manifold which, following Albert, will be called gradient vector fields. In particular, it is our intention to point out some close similarities between gradient vector fields on a cosymplectic manifold and (global) Hamiltonian vector fields on

‡ Senior Research Associate at the National Fund for Scientific Research (Belgium).

a symplectic manifold (the latter being the gradient vector fields of the symplectic structure).

First of all, it will be shown that gradient vector fields can be characterized in terms of a Lagrangian submanifold of the tangent bundle with respect to an appropriate symplectic structure. This property is well known in symplectic geometry, mainly due to the work of Tulczyjew [8, 9], and is in fact common to all gradient vector fields with respect to a non-degenerate covariant 2-tensor field (not necessarily skew-symmetric). Moreover, this property naturally leads us to introduce the notion of a local gradient vector field, the cosymplectic analogue of the local Hamiltonian vector field.

Secondly, starting from Albert's reduction theorem for cosymplectic manifolds, we will demonstrate that for gradient vector fields with symmetry one can establish a reduction scheme which is similar to the well known reduction procedure for Hamiltonian systems with symmetry [10, 11].

In section 2 we briefly recall some definitions and properties related to cosymplectic structures. Section 3 is devoted to the description of gradient and local gradient vector fields in terms of Lagrangian submanifolds of the tangent bundle. In section 4 we investigate the symmetry and reduction properties of gradient vector fields. This analysis is then further extended to the case of local gradient vector fields in section 5. We conclude in section 6 with a few additional remarks.

Throughout this paper, all manifolds, maps, vector fields and differential forms are assumed to be of class C^∞ . The sets of vector fields and differential forms over a manifold M are denoted by $\mathcal{X}(M)$ and $\bigwedge(M)$, respectively. For $X \in \mathcal{X}(M)$ and $\alpha \in \bigwedge(M)$ we write $i_X \alpha$ for the inner product (or also $\langle X, \alpha \rangle$, in case α is a 1-form) and $\mathcal{L}_X \alpha$ for the Lie derivative of α with respect to X . The tangent map of a map $f: M \rightarrow N$ will be denoted by $Tf: TM \rightarrow TN$.

2. Cosymplectic manifolds

A *cosymplectic manifold* is a triple (M, θ, ω) consisting of a smooth $(2n + 1)$ -dimensional manifold M with a closed 1-form θ and a closed 2-form ω , such that $\theta \wedge \omega^n \neq 0$ (see e.g. [7]). In particular, $\theta \wedge \omega^n$ yields a volume form on M . In the case where θ is exact we are precisely dealing with a canonical manifold with closed 2-form in the sense of [1, 5]. The standard example of a cosymplectic manifold is provided by an 'extended cotangent bundle' $(\mathbb{R} \times T^*N, dt, \pi^* \Omega_N)$, with $t: \mathbb{R} \times T^*N \rightarrow \mathbb{R}$ and $\pi: \mathbb{R} \times T^*N \rightarrow T^*N$ the canonical projections and Ω_N the canonical symplectic form on T^*N .

Let M be a manifold and let $\theta \in \bigwedge^1(M)$ and $\omega \in \bigwedge^2(M)$ be given, with $d\theta = 0$ and $d\omega = 0$. Consider the bundle homomorphism

$$\chi_{\theta, \omega}: TM \rightarrow T^*M \quad v \in T_x M \rightarrow \chi_{\theta, \omega}(v) = i_v \omega(x) + (i_v \theta(x))\theta(x). \quad (2.1)$$

Then we have the following important characterization of cosymplectic (and symplectic) manifolds.

Proposition 1 (cf [7]). (i) If (M, θ, ω) is cosymplectic, then $\chi_{\theta, \omega}$ is a smooth vector bundle isomorphism.

(ii) If $\chi_{\theta, \omega}$ is a smooth vector bundle isomorphism, then either (M, θ, ω) is a cosymplectic manifold, in which case M is odd dimensional, or (M, ω) is a symplectic manifold, in which case M is even dimensional.

According to this proposition it is clear that geometrical objects and properties related to a structure (M, θ, ω) , with θ a closed 1-form and ω a closed 2-form on M such that $\chi_{\theta, \omega}$ is an isomorphism, yield a cosymplectic (i.e. odd-dimensional) and a symplectic (i.e. even-dimensional) transcription. In the following we will confine ourselves to the cosymplectic case. For the following definitions and properties on cosymplectic manifolds, we again refer to [7].

Let (M, θ, ω) be a $(2n + 1)$ -dimensional cosymplectic manifold. On M there exists a distinguished vector field R , the Reeb vector field, defined by

$$i_R \theta = 1 \qquad i_R \omega = 0 \qquad (2.2)$$

i.e. $R = \chi_{\theta, \omega}^{-1} \circ \theta$ in terms of the bundle isomorphism (2.1). The manifold M admits an atlas of canonical ('Darboux') charts: in the neighbourhood of every point one can determine canonical coordinates (t, q^i, p_i) , $i = 1, \dots, n$, such that

$$\theta = dt \qquad \omega = dq^i \wedge dp_i. \qquad (2.3)$$

In terms of canonical coordinates, the Reeb vector field (2.2) reads $R = \partial/\partial t$.

By means of the isomorphism $\chi_{\theta, \omega}$ one can associate with every function $f \in C^\infty(M)$ a vector field $\text{grad } f$ on M , called the *gradient vector field*, which is defined by

$$\text{grad } f = \chi_{\theta, \omega}^{-1} \circ df. \qquad (2.4)$$

Equivalently, one has (cf [7])

$$i_{\text{grad } f} \theta = R(f) \qquad i_{\text{grad } f} \omega = df - R(f)\theta. \qquad (2.5)$$

In canonical coordinates we find

$$\text{grad } f = \frac{\partial f}{\partial t} \frac{\partial}{\partial t} + \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}.$$

On $C^\infty(M)$ one can define a Poisson bracket by

$$\{f, g\} = \omega(\text{grad } f, \text{grad } g). \qquad (2.6)$$

This induces a Poisson structure on (M, θ, ω) , the symplectic leaves of which are precisely the leaves of the integrable distribution $\ker \theta$. With every $f \in C^\infty(M)$ one can also associate a *Hamiltonian vector field* X_f according to

$$X_f = \chi_{\theta, \omega}^{-1} \circ (df - R(f)\theta)$$

or, equivalently,

$$i_{X_f} \theta = 0 \qquad i_{X_f} \omega = df - R(f)\theta. \qquad (2.7)$$

In canonical coordinates X_f reads

$$X_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}.$$

One can show that the assignment $C^\infty(M) \rightarrow \mathcal{X}(M)$, $f \rightarrow X_f$ is a Lie algebra anti-homomorphism with respect to the Poisson bracket (2.6) and the commutator of vector fields, i.e.

$$X_{\{f,g\}} = -[X_f, X_g] \tag{2.8}$$

(see [7]). From (2.5) and (2.7) it is readily seen that

$$i_{X_f} \omega = i_{\text{grad } f} \omega \quad \text{for all } f \in C^\infty(M)$$

and

$$X_f = \text{grad } f \quad \text{iff} \quad R(f) = 0.$$

The vector field $E_f = R + X_f$ is sometimes called the *evolution vector field* corresponding to f [5]. In canonical coordinates we recognize the expression of a time-dependent Hamiltonian vector field, i.e.

$$E_f = \frac{\partial}{\partial t} + \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}.$$

An *automorphism* of the cosymplectic structure (M, θ, ω) is a diffeomorphism $\phi : M \rightarrow M$ such that

$$\phi^* \theta = \theta \quad \phi^* \omega = \omega.$$

These automorphisms, also called ‘strong’ automorphisms, form a subgroup of the group of ‘weak’ automorphisms which are characterized by $\phi^* \theta = \theta$ and $\phi^* \omega = \omega - dh_\phi \wedge \theta$ for some $h_\phi \in C^\infty(M)$. The weak automorphisms correspond to the global canonical transformation in the sense of Lichnerowicz [1]. A (strong) *infinitesimal automorphism* of the cosymplectic structure is a vector field X on M such that

$$\mathcal{L}_X \theta = 0 \quad \mathcal{L}_X \omega = 0. \tag{2.9}$$

The infinitesimal automorphisms constitute a Lie subalgebra of $\mathcal{X}(M)$. Note that a Hamiltonian vector field X_f , as well as the corresponding evolution vector field E_f , will be an infinitesimal automorphism iff $\theta \wedge dR(f) = 0$ and thus, in particular, if $R(f) = 0$ (i.e. if f is ‘time independent’). It can be shown, however, that the Hamiltonian vector fields determine an ideal of the Lie algebra of weak infinitesimal automorphisms, the latter being characterized by $\mathcal{L}_X \theta = 0$ and $\mathcal{L}_X \omega = \theta \wedge dh_X$ for some $h_X \in C^\infty(M)$ (cf [7], proposition 3).

3. Gradient vector fields and Lagrangian submanifolds

Given a cosymplectic manifold (M, θ, ω) , the map $\chi_{\theta, \omega}$ defined by (2.1) is a vector bundle isomorphism over the identity from TM to T^*M . One can see that $\chi_{\theta, \omega}$ is in fact induced by the non-degenerate covariant 2-tensor field $\omega + \theta \otimes \theta$ on M . Let $\Omega_M = -d\Theta_M$ denote the canonical symplectic form on T^*M , with Θ_M the canonical

(or Liouville) 1-form. By means of the isomorphism $\chi_{\theta, \omega}$ one can pull-back Ω_M to TM , i.e.

$$\Omega_0 = \chi_{\theta, \omega}^* \Omega_M$$

and Ω_0 clearly yields a symplectic form on TM . We will derive another expression for Ω_0 , explicitly in terms of θ and ω . Before doing so we first make a small digression into the theory of derivations of forms.

Let N be an arbitrary smooth manifold. Following Tulczyjew, one can define two derivations, i_T and d_T from $\wedge(N)$ into $\wedge(TN)$ of degree -1 and 0 , respectively (see e.g. [8]). Given a p -form α on N , $i_T \alpha$ is a $(p-1)$ -form on TN defined by

$$i_T \alpha(v)(w_1, \dots, w_{p-1}) = \alpha(x)(v, u_1, \dots, u_{p-1}) \tag{3.1}$$

where $v \in T_x N$, $w_i \in T_v(TN)$ and $u_i = T\tau_N(w_i)$ for $i = 1, \dots, p-1$, with $\tau_N : TN \rightarrow N$ the tangent bundle projection. In particular, $i_T f = 0$ for functions (by convention) and if α is a 1-form, $i_T \alpha(v) = \langle v, \alpha(\tau_N(v)) \rangle$ where, as usual, $\langle \cdot, \cdot \rangle$ denotes the natural pairing between vectors and covectors. If $\alpha \in \wedge^1(N)$ is locally represented by $\alpha = \alpha_i(q) dq^i$, then, in natural bundle coordinates (q^i, v^i) on TN , $i_T \alpha = \alpha_i(q) v^i$. The derivative operator $d_T : \wedge(N) \rightarrow \wedge(TN)$ is defined by

$$d_T = i_T d + di_T.$$

Alternatively, one can say that i_T and d_T are derivations along τ_N of type i_* and d_* , respectively, induced by the vector field T along τ_N which locally reads $T = v^i \partial / \partial q^i$ (see e.g. [12]). Furthermore, it should be noticed that for $\alpha \in \wedge(N)$, $d_T \alpha$ precisely corresponds to the *complete lift* of α to TN in the sense of Yano and Ishihara [13], the latter being defined as follows. Let X be a vector field on N with (local) flow $\{\phi_t\}$. Then $\{T\phi_t\}$ defines a (local) flow on TN and its infinitesimal generator $X^c \in \mathcal{X}(TN)$ is called the complete lift of X . In coordinates, if $X = X^i \partial / \partial q^i$, then $X^c = X^i \partial / \partial q^i + (\partial X^i / \partial q^j) v^j \partial / \partial v^i$. The complete lift of a function $f \in C^\infty(N)$ is the function f^c on TN defined by $f^c = p_2 \circ Tf$ where $p_2 : T\mathbb{R} \cong \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is the projection onto the second factor. Let now α be an arbitrary p -form on N , then its complete lift to TN is the p -form α^c which is fully characterized by the property

$$\alpha^c(X_1^c, \dots, X_p^c) = (\alpha(X_1, \dots, X_p))^c$$

for any $X_i \in \mathcal{X}(N)$ (cf [13]). One can then easily verify, using local coordinate expressions for instance, that

$$\alpha^c = d_T \alpha.$$

We now return to the cosymplectic manifold (M, θ, ω) . The natural projections of TM and T^*M on M will be denoted by τ_M and π_M , respectively, and we recall that Θ_M stands for the canonical Liouville 1-form on T^*M .

Proposition 2. It holds that

$$\chi_{\theta, \omega}^* \Theta_M = i_T \omega + (i_T \theta) \tau_M^* \theta. \tag{3.2}$$

Proof. We will prove this relation pointwise. For any $w \in T(TM)$, with $v = \tau_{TM}(w)$ and $\eta = \chi_{\theta,\omega}(v)$, we have

$$\begin{aligned} \langle w, (\chi_{\theta,\omega}^* \Theta_M)(v) \rangle &= \langle T\chi_{\theta,\omega}(w), \Theta_M(\eta) \rangle \\ &= \langle T\pi_M(T\chi_{\theta,\omega}(w)), \tau_{T^*M}(T\chi_{\theta,\omega}(w)) \rangle \end{aligned}$$

where the second equality follows from the defining property of the Liouville 1-form (see e.g. [10]). Now, $\tau_{T^*M} \circ T\chi_{\theta,\omega} = \chi_{\theta,\omega} \circ \tau_{TM}$ and $\chi_{\theta,\omega}$ being a bundle isomorphism over the identity, we also have that $\pi_M \circ \chi_{\theta,\omega} = \tau_M$. Using the definition (2.1) of $\chi_{\theta,\omega}$, the previous relation then becomes

$$\begin{aligned} \langle w, (\chi_{\theta,\omega}^* \Theta_M)(v) \rangle &= \langle T\tau_M(w), \chi_{\theta,\omega}(\tau_{TM}(w)) \rangle \\ &= \omega(x)(\tau_{TM}(w), T\tau_M(w)) \\ &\quad + \langle \tau_{TM}(w), \theta(x) \rangle \langle T\tau_M(w), \theta(x) \rangle \end{aligned}$$

with $x = \tau_M(v)$. Finally, taking into account the definition (3.1) of the derivative operator i_T , we obtain

$$\begin{aligned} \langle w, (\chi_{\theta,\omega}^* \Theta_M)(v) \rangle &= \langle w, (i_T\omega)(v) \rangle + i_T\theta(v) \langle w, (\tau_M^*\theta)(v) \rangle \\ &= \langle w, (i_T\omega + (i_T\theta)(v)\tau_M^*\theta)(v) \rangle. \end{aligned}$$

Since this holds for any $w \in T(TM)$, the relation (3.2) readily follows. □

Taking the exterior derivative of both sides of (3.2) and recalling that both ω and θ are closed, we immediately obtain, with $\Omega_M = -d\Theta_M$:

Corollary 1. The symplectic form $\Omega_0 = \chi_{\theta,\omega}^* \Omega_M$, induced on the tangent bundle TM of a cosymplectic manifold (M, θ, ω) , reads

$$\Omega_0 = -(d_T\omega + d_T\theta \wedge \tau_M^*\theta). \tag{3.3}$$

Taking into account the above remark about the complete lift of differential forms and putting $\theta^v = \tau_M^*\theta$ (i.e. the vertical lift of θ), (3.3) can still be rewritten as

$$\Omega_0 = -(\omega^c + \theta^c \wedge \theta^v).$$

This expression agrees with the one derived by Aniculăesei [14] for the structure of the tangent bundle of almost contact and almost cosymplectic manifolds. Let us denote the natural bundle coordinates on TM corresponding to a canonical chart (t, q^i, p_i) on M by $(t, q^i, p_i, u, v^i, w_i)$. Using the expressions (2.3) for θ and ω , we find the following coordinate expression for Ω_0 :

$$\Omega_0 = dp_i \wedge dv^i - dq^i \wedge dw_i + dt \wedge du.$$

By construction $\chi_{\theta,\omega}$ is a symplectomorphism from (TM, Ω_0) to (T^*M, Ω_M) . Moreover, one can infer from the definition (2.4) of a gradient vector field that for any $f \in C^\infty(M)$ the image of $\text{grad } f$ is the pre-image under $\chi_{\theta,\omega}$ of the Lagrangian submanifold $\text{im}(df)$ of (T^*M, Ω_M) . This immediately implies:

Proposition 3. For each $f \in C^\infty(M)$, $\text{im}(\text{grad } f)$ is a Lagrangian submanifold of (TM, Ω_0) .

(Recall that a Lagrangian submanifold of a $2n$ -dimensional symplectic manifold (N, Ω) is a n -dimensional submanifold L of N such that the restriction of Ω to L vanishes. See, for instance, [10, 11] for more details.)

Note that in case of a symplectic manifold (M, ω) , with $\theta \equiv 0$, all the above reduces to the well known results about the induced symplectic structure on TM and the characterization of Hamiltonian vector fields as Lagrangian sections of TM (cf [8, 9]). As a matter of fact, the whole analysis applies to any manifold M equipped with a non-degenerate covariant 2-tensor field or, equivalently, a vector bundle isomorphism over the identity between TM and T^*M , together with the corresponding concept of gradient vector field.

The converse of proposition 3 does not hold in general, i.e. if a vector field X on a cosymplectic manifold determines a Lagrangian section of the tangent bundle, it need not be true that X is a gradient vector field. Again referring to the analogous situation in the symplectic case, we are naturally led to consider the notion of local gradient vector field. A vector field X on a cosymplectic manifold (M, θ, ω) is called a *local gradient vector field* if $\chi_{\theta, \omega} \circ X$ yields a closed 1-form, i.e.

$$d(i_X \omega + (i_X \theta)\theta) = 0$$

or, equivalently, since both ω and θ are closed,

$$\mathcal{L}_X \omega = \theta \wedge \mathcal{L}_X \theta. \tag{3.4}$$

Gradient vector fields are obviously local gradient vector fields. The set of local gradient vector fields furthermore contains the Lie algebra of infinitesimal automorphisms (2.9). Although the local gradient vector fields do not form a Lie algebra, one can easily verify that the Lie bracket of an infinitesimal automorphism X and a local gradient vector field Y is again a local gradient vector field. Indeed, using (2.9) and (3.4) we find that

$$\begin{aligned} \mathcal{L}_{[X, Y]} \omega &= \mathcal{L}_X \mathcal{L}_Y \omega \\ &= \mathcal{L}_X (\theta \wedge \mathcal{L}_Y \theta) \\ &= \theta \wedge \mathcal{L}_{[X, Y]} \theta. \end{aligned}$$

We now arrive at the main result of this section, from which proposition 3 can in fact be derived as a corollary.

Theorem 1. A vector field X on a cosymplectic manifold (M, θ, ω) is a local gradient vector field iff $\text{im}(X)$ is a Lagrangian submanifold of (TM, Ω_0) .

Proof. Knowing that $\chi_{\theta, \omega}$ is a symplectic diffeomorphism, the result immediately follows from the definition of local gradient vector fields and the well known property that the image of a 1-form on M is a Lagrangian submanifold of (T^*M, Ω_M) iff that 1-form is closed. □

This theorem is the cosymplectic analogue of the characterization of local Hamiltonian vector fields on symplectic manifolds (cf [9]).

4. Symmetry and reduction of gradient vector fields

In this section we will be dealing with smooth left actions $\Phi : G \times M \rightarrow M$ of a Lie group G on a cosymplectic manifold (M, θ, ω) . It will always be tacitly assumed that both G and M are connected. The Lie algebra of G will be denoted by \mathcal{G} and its dual by \mathcal{G}^* . For each $g \in G$ we put $\Phi_g \equiv \Phi(g, \cdot)$, the induced transformation on M . The fundamental vector field, or infinitesimal generator, associated with $\xi \in \mathcal{G}$ is the vector field ξ_M on M defined by

$$\xi_M(x) = \left. \frac{d}{dt} \Phi(\exp t\xi, x) \right|_{t=0}.$$

An action Φ of a Lie group G on a cosymplectic manifold (M, θ, ω) is called an *automorphism action*, or G is said to be an automorphism group of (M, θ, ω) , if for each $g \in G$ the corresponding Φ_g is an automorphism of the cosymplectic structure, i.e.

$$\Phi_g^* \theta = \theta \qquad \Phi_g^* \omega = \omega.$$

It then follows that the fundamental vector fields are infinitesimal automorphisms, i.e.

$$\mathcal{L}_{\xi_M} \theta = 0 \qquad \mathcal{L}_{\xi_M} \omega = 0$$

for each $\xi \in \mathcal{G}$. An automorphism action Φ will be called a *restricted Hamiltonian action* if furthermore, for each $\xi \in \mathcal{G}$, the associated fundamental vector field ξ_M is a Hamiltonian vector field with Hamiltonian J_ξ satisfying $R(J_\xi) = 0$, i.e.

$$i_{\xi_M} \theta = 0 \qquad i_{\xi_M} \omega = dJ_\xi. \tag{4.1}$$

In canonical coordinates (t, q^i, p_i) , this still means that the functions J_ξ do not depend on the ‘time’ coordinate t . We use the denomination ‘restricted’ Hamiltonian action to distinguish it from the more general Hamiltonian actions on canonical manifolds studied by Marle [5], where no time independence of the Hamiltonians J_ξ is required. We can always arrange things such that the map $\xi \rightarrow J_\xi$ is \mathbb{R} -linear.

With a restricted Hamiltonian action one can associate a momentum map $J : M \rightarrow \mathcal{G}^*$ defined in the usual way by

$$\langle \xi, J(x) \rangle = J_\xi(x)$$

for all $\xi \in \mathcal{G}$. The non-uniqueness of the Hamiltonians J_ξ is reflected in the non-uniqueness of the momentum map. In [7] it has been shown that there always exists an affine action Ψ of G on \mathcal{G}^* such that J is equivariant with respect to Φ and Ψ , i.e.

$$J \circ \Phi_g = \Psi_g \circ J$$

for each $g \in G$. The affine action Ψ is of the form

$$\Psi_g(\mu) = \text{Ad}_g^* \mu + \sigma(g) \qquad \mu \in \mathcal{G}^*$$

with Ad^* the co-adjoint representation of G on \mathcal{G}^* and σ a (non-homogeneous) 1-cocycle of G with values in \mathcal{G}^* , given by $\sigma(g) = J(\Phi_g(x)) - \text{Ad}_g^*(J(x))$. The cohomology class of σ only depends on the given action Φ and not on the particular choice of the momentum map.

For given $\mu \in \mathcal{G}^*$ we denote by G_μ the isotropy group of μ with respect to the affine action Ψ . By equivariance of J it follows that $J^{-1}\{\mu\}$ is an invariant subset for the restriction of Φ to G_μ . Moreover, if μ is a weakly regular value (and thus, in particular, a regular value) of J , then $J^{-1}\{\mu\}$ is a submanifold of M and Φ induces a smooth action of G_μ on $J^{-1}\{\mu\}$. Following Libermann and Marle [11] we will say that this action is *simple* if the orbit space $J^{-1}\{\mu\}/G_\mu$ admits a smooth manifold structure such that the canonical projection $\pi_\mu : J^{-1}\{\mu\} \rightarrow J^{-1}\{\mu\}/G_\mu$ is a surjective submersion. This will for instance be the case if the action is free and proper [10]. In the following it will always be assumed that G_μ is connected such that the fibres of π_μ are also connected. (If this were not the case, one could simply restrict the further analysis to the connected component of the identity of G_μ .)

Albert has established the following cosymplectic reduction theorem which will play a central role in the further discussion.

Theorem 2 (cf [7]). Given a restricted Hamiltonian action of a Lie group G on a cosymplectic manifold (M, θ, ω) with momentum map J , let $\mu \in \mathcal{G}^*$ be a weakly regular value of J and assume the induced action of G_μ on $J^{-1}\{\mu\}$ is simple. Then the quotient space $\mathcal{P}_\mu = J^{-1}\{\mu\}/G_\mu$ admits a cosymplectic structure (θ_μ, ω_μ) such that

$$j_\mu^* \theta = \pi_\mu^* \theta_\mu \quad \text{and} \quad j_\mu^* \omega = \pi_\mu^* \omega_\mu$$

with $j_\mu : J^{-1}\{\mu\} \rightarrow M$ the inclusion map and $\pi_\mu : J^{-1}\{\mu\} \rightarrow \mathcal{P}_\mu$ the canonical projection.

The main purpose of this section now consists in proving that in the case where a restricted Hamiltonian action defines a symmetry of a gradient vector field, then the latter gives rise to a gradient vector field on the reduced cosymplectic manifold $(\mathcal{P}_\mu, \theta_\mu, \omega_\mu)$ corresponding to any weakly regular value μ of the momentum map. This then yields the analogue of the Marsden-Weinstein reduction of a Hamiltonian system with symmetry on a symplectic manifold [10, 11].

An action Φ of a Lie group G on a cosymplectic manifold (M, θ, ω) is said to be a *symmetry* of a gradient vector field, $\text{grad } f$, if for each $g \in G$

$$\Phi_g^*(\text{grad } f) = \text{grad } f.$$

This still implies (and, in view of the assumed connectedness of G , is equivalent to)

$$[\xi_M, \text{grad } f] = 0$$

for each $\xi \in \mathcal{G}$.

Proposition 4. Let Φ be an automorphism action of a Lie group G on a cosymplectic manifold (M, θ, ω) . Then Φ is a symmetry of $\text{grad } f$ iff for each $\xi \in \mathcal{G}$, $\xi_M(f) = c_\xi$ for some real constant c_ξ .

Proof. By definition for a gradient vector field

$$i_{\text{grad } f}\omega + (i_{\text{grad } f}\theta)\theta = df.$$

Taking the Lie derivative of both sides with respect to ξ_M , and taking into account $\mathcal{L}_{\xi_M}\theta = 0$ and $\mathcal{L}_{\xi_M}\omega = 0$, we obtain

$$i_{[\xi_M, \text{grad } f]}\omega + \left(i_{[\xi_M, \text{grad } f]}\theta\right)\theta = d\xi_M(f)$$

i.e.

$$\chi_{\theta, \omega} \circ [\xi_M, \text{grad } f] = d\xi_M(f).$$

Now, $\chi_{\theta, \omega}$ being an isomorphism and since M is assumed to be connected, it follows that

$$[\xi_M, \text{grad } f] = 0 \quad \text{iff} \quad \xi_M(f) = c_\xi$$

for some $c_\xi \in \mathbb{R}$. □

From this we immediately deduce the following.

Corollary 2. If an automorphism action Φ of G on a cosymplectic manifold (M, θ, ω) leaves invariant a function $f \in C^\infty(M)$, i.e. $\Phi_g^*f = f$ for all $g \in G$, then it is a symmetry of $\text{grad } f$.

The next proposition also provides a basic ingredient in establishing a reduction property of gradient vector fields with symmetry.

Proposition 5. Let $\Phi : G \times M \rightarrow M$ be a restricted Hamiltonian action with momentum map J . If Φ leaves invariant a function $f \in C^\infty(M)$, then J is a \mathcal{G}^* -valued first integral of $\text{grad } f$ (i.e. J is constant along the orbits of $\text{grad } f$).

Proof. According to (4.1), the fundamental vector fields ξ_M satisfy $i_{\xi_M}\omega = dJ_\xi$. Taking the contraction of both sides with $\text{grad } f$ we find, with (2.5),

$$\begin{aligned} \text{grad } f(J_\xi) &= -i_{\xi_M}i_{\text{grad } f}\omega \\ &= -i_{\xi_M}(df - R(f)\theta) \\ &= -\xi_M(f) + R(f)i_{\xi_M}\theta. \end{aligned}$$

In view of (4.1) and the assumed invariance of f , it follows that $\text{grad } f(J_\xi) = 0$. Since this holds for each $\xi \in \mathcal{G}$, we may indeed conclude that J is invariant under the flow of $\text{grad } f$. □

We can now state the following reduction theorem for gradient vector fields on cosymplectic manifolds.

Theorem 3. Let $\Phi : G \times M \rightarrow M$ be a restricted Hamiltonian action on a cosymplectic manifold (M, θ, ω) with momentum map J . Let $\mu \in \mathcal{G}^*$ be a weakly regular value of J and assume the induced action of G_μ on $J^{-1}\{\mu\}$ is simple such that the conditions of theorem 2 are verified. Then, if $f \in C^\infty(M)$ is invariant under Φ , $\text{grad } f$ is tangent to $J^{-1}\{\mu\}$ and its restriction to $J^{-1}\{\mu\}$ projects onto the reduced cosymplectic manifold $(\mathcal{P}_\mu, \theta_\mu, \omega_\mu)$. Moreover, its projection $(\text{grad } f)_\mu$ is again a gradient vector field, i.e.

$$(\text{grad } f)_\mu = \text{grad } f_\mu$$

for some $f_\mu \in C^\infty(\mathcal{P}_\mu)$ satisfying $f|_{J^{-1}\{\mu\}} = \pi_\mu^* f_\mu$.

Proof. Under the given assumptions it follows from proposition 5 that $J^{-1}\{\mu\}$ is an invariant submanifold of $\text{grad } f$. Moreover, according to corollary 2, the given action Φ is a symmetry of $\text{grad } f$. In particular, this implies that the induced action of G_μ on $J^{-1}\{\mu\}$ commutes with the flow of $\text{grad } f|_{J^{-1}\{\mu\}}$. Hence, the latter induces canonically a flow on \mathcal{P}_μ and we denote its infinitesimal generator by $(\text{grad } f)_\mu$. The vector fields $\text{grad } f|_{J^{-1}\{\mu\}}$ and $(\text{grad } f)_\mu$ are π_μ -related. It is then straightforward to verify that

$$(\text{grad } f)_\mu = \chi_{\theta_\mu, \omega_\mu}^{-1} \circ \text{d}f_\mu$$

with f_μ being uniquely determined by $f|_{J^{-1}\{\mu\}} = \pi_\mu^* f_\mu$, π_μ being a surjective submersion with connected fibres.

(The argument is completely similar to the one used in the reduction theorem for Hamiltonian systems with symmetry on symplectic manifolds [10, 11].) \square

Without going into details we will close this section by briefly sketching how the above reduction theorems 2 and 3 can be translated in terms of symplectic reductions, using the results of the previous section. For completeness, let us first recall that, given a symplectic manifold (N, Ω) , a *symplectic reduction* is defined as a surjective submersion $\pi : N_1 \rightarrow P$ from a submanifold N_1 of N onto another symplectic manifold $(P, \tilde{\Omega})$, such that $j_1^* \tilde{\Omega} = \pi^* \Omega$, with $j_1 : N_1 \rightarrow N$ the inclusion map (cf [11, 15]). Similarly, given a cosymplectic manifold (M, θ, ω) , we will say that a *cosymplectic reduction* is a surjective submersion $\pi : M_1 \rightarrow P$, from a submanifold M_1 of M onto another cosymplectic manifold $(P, \tilde{\theta}, \tilde{\omega})$, such that $j_1^* \tilde{\theta} = \pi^* \theta$ and $j_1^* \tilde{\omega} = \pi^* \omega$, with $j_1 : M_1 \rightarrow M$ the inclusion map. Under the hypotheses of theorem 2 we now have that $\pi_\mu : J^{-1}\{\mu\} \rightarrow \mathcal{P}_\mu$ is a cosymplectic reduction in the above sense. This clearly lifts to a symplectic reduction $T\pi_\mu : TJ^{-1}\{\mu\} \rightarrow T\mathcal{P}_\mu$, with $TJ^{-1}\{\mu\}$ a submanifold of the symplectic tangent bundle (TM, Ω_0) and $T\mathcal{P}_\mu$ being equipped with the symplectic form $\Omega_\mu = \chi_{\theta_\mu, \omega_\mu}^*(\Omega_{\mathcal{P}_\mu})$. If, moreover, the conditions of theorem 3 are verified, then we see that $\text{im}(\text{grad } f) \cap TJ^{-1}\{\mu\} = \text{im}(\text{grad } f|_{J^{-1}\{\mu\}})$, and this submanifold is mapped by the symplectic reduction $T\pi_\mu$ onto the Lagrangian submanifold $\text{im}(\text{grad } f_\mu)$ of $(T\mathcal{P}_\mu, \Omega_\mu)$.

5. Local gradient vector fields with symmetry

In this section we will study symmetries of local gradient vector fields on a cosymplectic manifold. The treatment is mainly inspired by the symmetry analysis of local Hamiltonian systems on symplectic manifolds of Cariñena and Ibort [16].

Let us consider again a restricted Hamiltonian action $\Phi : G \times M \rightarrow M$ on a cosymplectic manifold (M, θ, ω) . By definition, the fundamental vector fields of this action are Hamiltonian with Hamiltonian functions that are invariant under the flow of the Reeb vector field R . From section 2 we then know that these vector fields are also gradient vector fields, i.e. for each $\xi \in \mathcal{G}$

$$\xi_M = X_{J_\xi} = \text{grad } J_\xi$$

with $R(J_\xi) = 0$. On M there exists a Poisson structure $\{ , \}$ defined by (2.6). Hence, we see that for any $\xi, \eta \in \mathcal{G}$

$$\begin{aligned} \{J_\xi, J_\eta\} &= \omega(\xi_M, \eta_M) \\ &= \eta_M(J_\xi) = -\xi_M(J_\eta). \end{aligned} \tag{5.1}$$

The map $\mathcal{G} \rightarrow \mathcal{X}(M)$, $\xi \rightarrow \xi_M$ is a Lie algebra anti-homomorphism, i.e. $[\xi, \eta]_M = -[\xi_M, \eta_M]$. Computing the differential of $J_{[\xi, \eta]}$ we find, using (4.1) and (5.1):

$$\begin{aligned} dJ_{[\xi, \eta]} &= i_{[\xi, \eta]_M} \omega = -i_{[\xi_M, \eta_M]} \omega \\ &= -\mathcal{L}_{\xi_M} i_{\eta_M} \omega \\ &= -d\xi_M(J_\eta) \\ &= d\{J_\xi, J_\eta\}. \end{aligned}$$

Since M is still assumed to be connected it follows that

$$J_{[\xi, \eta]} = \{J_\xi, J_\eta\} + \Sigma(\xi, \eta) \tag{5.2}$$

for some real constant $\Sigma(\xi, \eta)$. In terms of the associated momentum map J this relation can still be rewritten as

$$\langle [\xi, \eta], J \rangle = \langle \xi, J \rangle, \langle \eta, J \rangle + \Sigma(\xi, \eta).$$

The map $\Sigma : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{R}$ is bilinear and skewsymmetric, and it is straightforward to verify that it is a real-valued 2-cocycle of \mathcal{G} .

Suppose the restricted Hamiltonian action Φ is a symmetry of a local gradient vector field X on M , i.e. $[\xi_M, X] = 0$ for each $\xi \in \mathcal{G}$, with X satisfying (3.4). We can then prove that the functions $X(J_\xi)$ are constant for all ξ and, as a matter of fact, the converse is also true.

Proposition 6. A restricted Hamiltonian action, with momentum map J , is a symmetry of a local gradient vector field X iff for each $\xi \in \mathcal{G}$ it holds that $X(J_\xi) = \ell_\xi$ for some constant $\ell_\xi \in \mathbb{R}$.

Proof. Let $X \in \mathcal{X}(M)$ be a local gradient vector field. Then, using (3.8) and (4.1) we see that

$$\begin{aligned} dX(J_\xi) &= \mathcal{L}_X i_{\xi_M} \omega \\ &= i_{[X, \xi_M]} \omega + i_{\xi_M} \mathcal{L}_X \omega \\ &= i_{[X, \xi_M]} \omega + \left(i_{[X, \xi_M]} \theta \right) \theta \\ &= \chi_{\theta, \omega} \circ [X, \xi_M]. \end{aligned}$$

Consequently, since $\chi_{\theta, \omega}$ is a bundle isomorphism, $[X, \xi_M] = 0$ iff $dX(J_\xi) = 0$, from which the result follows. □

Given any $\xi, \eta \in \mathcal{G}$ one easily finds, using (5.1) and (5.2), and the previous proposition, that

$$\begin{aligned} \ell_{[\xi, \eta]} &= \mathcal{L}_X(J_{[\xi, \eta]}) \\ &= \mathcal{L}_X(\{J_\xi, J_\eta\}) \\ &= \mathcal{L}_{[X, \eta_M]}(J_\xi) + \eta_M(\ell_\xi) \\ &= 0 \end{aligned}$$

since $[X, \eta_M] = 0$, by assumption, and ℓ_ξ is a constant. From this we infer that in the case where \mathcal{G} is such that $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, in particular if G is a semisimple Lie group, the constants ℓ_ξ in the above proposition are all zero. We may therefore conclude with the following theorem.

Theorem 4. If a restricted Hamiltonian action $\Phi : G \times M \rightarrow M$ of a semisimple Lie group G on a cosymplectic manifold (M, θ, ω) , with momentum map J , is a symmetry of a local gradient vector field X , then J is a \mathcal{G}^* -valued first integral of X . If, furthermore, the conditions of theorem 2 are verified, then for any weakly regular value μ of J , X is tangent to $J^{-1}\{\mu\}$ and projects onto the reduced cosymplectic manifold $(\mathcal{P}_\mu, \theta_\mu, \omega_\mu)$. The projection is again a local gradient vector field.

Proof. The proof proceeds along the same lines as those of proposition 5 and theorem 3. □

6. Some final remarks

In this paper we have been dealing with a class of dynamical systems which, in appropriate coordinates, can be represented by a system of $2n + 1$ first-order ordinary differential equations of the form

$$\frac{dq^i}{ds} = \frac{\partial f}{\partial p_i} \quad \frac{dp_i}{ds} = -\frac{\partial f}{\partial q^i} \quad \frac{dt}{ds} = \frac{\partial f}{\partial t}$$

for some smooth function $f(t, q, p)$. In several respects these systems reveal a striking resemblance to Hamiltonian systems on symplectic manifolds, at least from a geometrical point of view. From the point of view of theoretical mechanics it must be conceded that, if t is to be regarded as the physical time coordinate, it is the evolution type vector fields $E_f = R + X_f$ rather than the gradient vector fields that are of particular interest. From the definitions in section 2 we see that

$$\text{grad } f - E_f = (R(f) - 1)R \tag{6.1}$$

and so the two only coincide in the special case where $R(f) = 1$. As far as the symmetry and reduction properties are concerned one may observe, however, that the treatment of section 4 also applies to evolution vector fields. Indeed, it has already been shown by Marle [5] that for group actions of the type considered in this paper (with $R(J) = 0$), which leave a function f invariant, the momentum map J will be

constant along the orbits of E_f . It is also straightforward to check, using (6.1) and the results of section 4, that such an action will be a symmetry of E_f . Hence, all ingredients are available to establish a reduction scheme for evolution vector fields with symmetry along the lines of theorem 3.

Finally, we still note that the characterization of (local) gradient vector fields in terms of Lagrangian submanifolds (cf section 3) can be extended to the case of gradient systems with constraints, following the analogous treatment of Hamiltonian systems with constraints as, for instance, in [8, 9].

Acknowledgments

This work was partially supported by a Research Grant from the National Fund for Scientific Research (Belgium) and by DGICYT-Spain, Proyecto PB88-0012. EAL also acknowledges invitations to the Consejo Superior de Investigaciones Científicas (Madrid) and the Departamento de Matemática Fundamental (La Laguna, Tenerife) where part of this work was conceived.

References

- [1] Lichnerowicz A 1976 *Topics in Differential Geometry* ed H Rund and W Forbes (New York: Academic) pp 57-85
- [2] Lichnerowicz A 1979 *Bull. Soc. Math. Belgique* **31** 105-35
- [3] Hamoui A and Lichnerowicz A 1984 *J. Math. Phys.* **25** 923-31
- [4] Flato M, Lichnerowicz A and Sternheimer D 1975 *J. Math. Pures et Appl.* **54** 445-80
- [5] Marle C-M 1983 *Symplectic Geometry (Research Notes in Mathematics)* ed A Crumeyrolle and J Grifone (Boston: Pitman) pp 144-66
- [6] Libermann P 1959 *Colloque de Géométrie Différentielle Globale (Bruxelles 1958)* (Paris: Gauthier-Villars) pp 37-59
- [7] Albert C 1989 *J. Geom. Phys.* **6** 627-49
- [8] Tulczyjew W M 1976 *C.R. Acad. Sci. Paris A* **283** 15-18
- [9] Menzio M R and Tulczyjew W M 1978 *Ann. Inst. H. Poincaré A* **28** 349-67
- [10] Abraham R and Marsden J 1978 *Foundations of Mechanics* 2nd edition (New York: Benjamin/Cummings)
- [11] Libermann P and Marle C-M 1987 *Symplectic Geometry and Analytical Mechanics* (Dordrecht: Reidel)
- [12] Saunders D 1987 *The Geometry of Jet Bundles (London Math. Soc. Lecture Notes Series)* **142** (Cambridge: Cambridge University Press)
- [13] Yano K and Ishihara S 1972 *Tangent and Cotangent Bundles* (New York: Marcel Dekker)
- [14] Aniculăseși G 1980 *Rev. Roum. Math. Pures et Appl.* **25** 995-1002
- [15] Benenti S 1983 *Proc. Int. Meeting on Geometry and Physics* ed M Modugno (Bologna: Pitagora) pp 11-41
- [16] Cariñena J F and Ibort L A 1985 *Nuovo Cimento B* **87** 41-8